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FRACTIONAL DIFFERENTIAL EQUATIONS: α-ENTIRE SOLUTIONS, REGULAR AND IRREGULAR SINGULARITIES

Anatoly N. Kochubei *

Abstract

We consider fractional differential equations of order $\alpha \in (0,1)$ for functions of one independent variable $t \in (0,\infty)$ with the Riemann-Liouville and Caputo-Dzhrbashyan fractional derivatives. A precise estimate for the order of growth of α-entire solutions is given. An analog of the Frobenius method for systems with regular singularity is developed. For a model example of an equation with a kind of an irregular singularity, a series for a formal solution is shown to be convergent for $t > 0$ (if α is an irrational number poorly approximated by rational ones) but divergent in the distribution sense.

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Key Words and Phrases: fractional differential equation; Riemann-Liouville derivative; Caputo-Dzhrbashyan derivative; regular singularity; irregular singularity

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1. Introduction

Fractional differential equations are widely used for modeling anomalous relaxation and diffusion phenomena; see [3, 12] for further references. Meanwhile the mathematical theory of such equations is still in its initial stage. In particular, a systematic development of the analytic theory of fractional differential equations with variable coefficients was initiated only recently, in the paper by Kilbas, Rivero, Rodríguez-Germá, and Trujillo [11] (see also Section 7.5 in [12]). For equations of order $\alpha \in (0,1)$, of the form

$$\left(\mathrm{D}^{\alpha}_{0+} u\right)(t) = a(t)u(t), \quad t > 0, \tag{1}$$

where D^{α}_{0+} is the Riemann-Liouville fractional derivative, or

$$\left(\mathbb{D}^{(\alpha)} u\right)(t) = a(t)u(t), \quad t > 0, \tag{2}$$

where $\mathbb{D}^{(\alpha)}$ is the Caputo-Dzhrbashyan fractional derivative, their main results are as follows. If $a(t) = A(t^{\alpha})$, and $A(z)$ is a real function possessing an absolutely convergent Taylor expansion on an interval $|z| < \theta$, then the equation (1) possesses a solution of the form

$$u(t) = t^{\alpha-1}\sum_{n=0}^{\infty} a_n t^{\alpha n}, \quad 0 < t < \theta,$$

while the equation (2) has a solution

$$u(t) = \sum_{n=0}^{\infty} b_n t^{\alpha n}, \quad 0 \le t < \theta.$$

In both cases the solutions are unique, if appropriate initial conditions are prescribed.

Thus, for example, the property of α-analyticity of the coefficient $a(t)$ (defined above) implies a similar property of a solution of the equation (2). In fact, we have $u(t) = U(t^{\alpha})$, where U is holomorphic in a disk $\{z \in \mathbb{C},\ |z| < \theta\}$. The coefficient a may be complex-valued as well.

The above results open the way for developing a theory of α-analytic solutions of fractional differential equations in the spirit of classical analytic theory of ordinary differential equations. Here we give some results in this direction.

If in (2) $a(t) = A(t^\alpha)$ where A is an entire function, then the above results from [11] with $\theta = \infty$ (stated there in a weaker form, only for real arguments of analytic functions, than actually proved) show that the solution of the Cauchy problem for the equation (2) is of the form $u(t) = U(t^\alpha)$, where U is an entire function. Following [11, 12] we call such solutions α-entire. In particular, that is true, if A is a polynomial. A natural question is about the order of U (here we investigate this subject just for the equation (2) since its properties are closer to those of ordinary differential equations). It is known (see, for example, [2] or [8]) that every nontrivial solution of the equation $u^{(k)}(z) = A(z)u(z)$, $k \in \mathbb{N}$, with a polynomial coefficient A, is an entire function of order $1 + \deg(A)/k$. In this paper we prove that the orders of the entire functions U corresponding to solutions of (2) do not exceed $(1 + \deg(A))/\alpha$. As $\alpha \to 1$, this agrees with the above differential equation result. On the other hand, if $\deg(A) = 0$, that is $A(z) = \lambda$, $\lambda \in \mathbb{C}$, then $U(z) = E_\alpha(\lambda z)$, where E_α is the Mittag-Leffler function whose order is $1/\alpha$ [3, 12], which shows the exactness of our general estimate.

Next, we investigate systems of fractional equations with regular singularity, that is the equations

$$t^\alpha \left(\mathrm{D}^\alpha_{0+} u\right)(t) = A(t^\alpha)u(t) \tag{3}$$

and

$$t^\alpha \left(\mathbb{D}^{(\alpha)} u\right)(t) = A(t^\alpha)u(t), \tag{4}$$

where $A(z)$ is a holomorphic matrix-function. Under some assumptions, we prove that formal power series solutions of (3) and (4) converge near the origin and develop an analog of the classical Frobenius method of finding a solution. For scalar equations, the latter problem was considered in [12, 15].

Finally, in order to clarify characteristic features of fractional equations with irregular singularity, we study a model example, the equation

$$t^{2\alpha} \left(\mathbb{D}^{(\alpha)} u\right)(t) = \lambda u(t), \quad \lambda \in \mathbb{C}, \tag{5}$$

where, as before, $0 < \alpha < 1$. Assuming that α is irrational and satisfies a Diophantine condition (that is α is poorly approximated by rational numbers), we construct a kind of a formal solution of (5) convergent for $t > 0$. We prove that the series for the formal solution does not converge in the distribution sense, within a theory of distributions associated with the fractional calculus (see [16, 18]). Thus, the formal solution $u(t)$ cannot be interpreted as a distribution solution. It is interesting that $u(t)$ is closely connected with a class of analytic functions with irregular behavior introduced by Hardy [6].

2. Preliminaries

2.1. Fractional derivatives and integrals [3, 12, 18]. Let $\alpha \in (0,1)$ be a fixed number. *The Riemann-Liouville fractional integral* of order α of a function $\varphi \in L_1(0,T)$ is defined as

$$\left(I_{0+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)\,d\tau, \quad 0 < t \le T.$$

The Riemann-Liouville fractional derivative of order α is given by the expression

$$\left(D_{0+}^{\alpha}\varphi\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha}\varphi(\tau)\,d\tau, \quad 0 < t \le T,$$

that is $\left(\mathrm{D}_{0+}^{\alpha}\varphi\right)(t) = \frac{d}{dt}\left(I_{0+}^{1-\alpha}\varphi\right)(t)$, provided the fractional integral $I_{0+}^{1-\alpha}\varphi$ is an absolutely continuous function. If φ is defined on the whole half-axis $(0,\infty)$, then $I_{0+}^{\alpha}\varphi$ and $\mathrm{D}_{0+}^{\alpha}\varphi$ are also defined on $(0,\infty)$. Below we will consider just this case.

Note that the Riemann-Liouville derivative is defined for some functions with a singularity at the origin. For example, if $\varphi(t) = t^d$, $d > -1$, then

$$\left(\mathrm{D}_{0+}^{\alpha}\varphi\right)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1-\alpha)} t^{d-\alpha}, \tag{6}$$

so that $\mathrm{D}_{0+}^{\alpha}\varphi = 0$, if $\varphi(t) = t^{\alpha-1}$. For $\varphi(t) = t^d$, $d > -1$, we have also

$$\left(I_{0+}^{\alpha}\varphi\right)(t) = \frac{\Gamma(d+1)}{\Gamma(d+1+\alpha)} t^{d+\alpha}. \tag{7}$$

The Riemann-Liouville fractional differentiation and integration are inverse to each other in the following sense. If $\varphi \in L_1(0,T)$, then $\mathrm{D}_{0+}^{\alpha} I_{0+}^{\alpha}\varphi = \varphi$. The equality $I_{0+}^{\alpha}\mathrm{D}_{0+}^{\alpha}\varphi = \varphi$ holds under the stronger assumption that $\varphi = I_{0+}^{\alpha}\psi$ with some $\psi \in L_1(0,T)$. The latter is equivalent to the conditions of absolute continuity of $I_{0+}^{1-\alpha}\varphi$ on $[0,T]$ and the equality $\left(I_{0+}^{1-\alpha}\varphi\right)(0) = 0$.

Let a function φ be continuous on $[0,T]$ and possess the Riemann-Liouville fractional derivative of order α. The function

$$\left(\mathbb{D}^{(\alpha)}\varphi\right)(t)=\frac{1}{\Gamma(1-\alpha)}\left[\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}\varphi(\tau)\,d\tau - t^{-\alpha}\varphi(0)\right]$$
$$=\left(\mathrm{D}^{\alpha}_{0+}\varphi\right)(t)-\frac{1}{t^{\alpha}\Gamma(1-\alpha)}\varphi(0) \quad (8)$$

is called *the Caputo-Dzhrbashyan, or regularized, fractional derivative.* If φ is absolutely continuous on $[0,T]$, then

$$\left(\mathbb{D}^{(\alpha)}\varphi\right)(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\varphi'(\tau)\,d\tau. \quad (9)$$

In contrast to D^{α}_{0+}, $\mathbb{D}^{(\alpha)}$ is defined only on continuous functions and vanishes on constant functions. In most of physical applications, equations with $\mathbb{D}^{(\alpha)}$ are used, because a solution of an equation with the Riemann-Liouville derivative typically has a singularity at the origin $t=0$, so that the initial state of a system to be described by the equation is not defined.

Let $v(z)=\sum\limits_{n=0}^{\infty} c_n z^n$ be an entire function. Consider the Caputo-Dzhrbashyan derivative of the function

$$\varphi(t)=v(t^{\alpha})=\sum_{n=0}^{\infty} c_n t^{\alpha n}.$$

It follows from (6) and (8) that

$$\left(\mathbb{D}^{(\alpha)}\varphi\right)(t)=\sum_{n=1}^{\infty} c_n\beta(n)t^{\alpha(n-1)}=\left(\mathfrak{D}_{\alpha}v\right)(t^{\alpha})$$

where

$$\beta(n)=\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha+1-\alpha)}, \quad (10)$$

and the operator

$$\left(\mathfrak{D}_{\alpha}v\right)(z)=\sum_{n=1}^{\infty} c_n\beta(n)z^{n-1}$$

is known as the Gelfond-Leontiev (G-L) operator of generalized differentiation (see [13, pp. 72-85], [18, pp. 426-427]; in fact, $\mathfrak{D}_{\alpha}$ is defined for wider classes of functions). As a matter of fact, this is a generalized G-L differentiation operator with respect to the Mittag-Leffler function (see e.g. in [13, 18]):

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)},\quad \alpha>0,$$

and for the G-L operators with respect to (arbitrary) entire function $\varphi(z)$, see the original work [4].

The operator, right inverse to $\mathfrak{D}_\alpha$, has the form

$$(\mathfrak{I}_\alpha f)(z)=\frac{z}{\Gamma(\alpha)}\int\limits_0^1 (1-t)^{\alpha-1}f(zt^\alpha)\,dt$$

(see Sect. 22.3 in [18]). It will be convenient to make a change of variables setting $z=Re^{i\theta}$, $r=Rt^\alpha$. Then

$$(\mathfrak{I}_\alpha f)(Re^{i\theta})=\frac{e^{i\theta}}{\alpha\Gamma(\alpha)}\int\limits_0^R\left[\left(\frac{R}{r}\right)^{1/\alpha}-1\right]^{\alpha-1}f(re^{i\theta})\,dr. \tag{11}$$

If $f(z)=\sum\limits_{k=0}^{\infty} f_k z^k$, then

$$(\mathfrak{I}_\alpha f)(z)=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1+\alpha)}f_k z^{k+1},$$

and it is easy to check that

$$(\mathfrak{I}_\alpha\mathfrak{D}_\alpha v)(z)=v(z)-v(0). \tag{12}$$

2.2. A class of distributions. Spaces of test functions and distributions behaving reasonably under the action of fractional integration operators were introduced by Rubin [16] (for a brief exposition see also [18]; both in [18] and [16] there are references regarding other approaches and earlier publications in this field). Proceeding from [16], it is easy to come to a class of distributions, where the Caputo-Dzhrbashyan derivative $\mathbb{D}^{(\alpha)}$ is defined in a natural way.

Let $\mathcal{S}(0,\infty)$ be the Schwartz space of smooth functions on $[0,\infty)$ with rapid decay at infinity. Denote

$$\mathcal{S}_+=\{\varphi\in\mathcal{S}(0,\infty):\ \varphi^{(l)}(0)=0,\ l=0,1,2,\dots\},$$

$$\Phi_+^{1-\alpha}=\left\{\varphi\in\mathcal{S}_+:\ \int\limits_0^\infty \varphi(x)x^{1-\alpha-k}\,dx=0,\ k=1,2,\dots\right\},$$

$$\Phi_+^{\alpha-1} = \left\{ \varphi \in \mathcal{S}_+ : \int\limits_0^\infty \varphi(x) x^{\alpha-1-k}\, dx = 0,\ k = 0, 1, 2, \dots \right\}.$$

These spaces are interpreted as topological vector spaces with the topologies induced from $\mathcal{S}(0,\infty)$; see [16] for various descriptions of these topologies including the description by seminorms.

Together with the Riemann-Liouville fractional integration operator I_{0+}^α, it is convenient to use the operator

$$\left(I_-^\alpha \varphi\right)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^\infty (\tau - t)^{\alpha-1} \varphi(\tau)\, d\tau, \quad t > 0.$$

If φ, ψ are sufficiently good functions, for example, if $\varphi \in L_p(0,\infty)$, $\psi \in L_q(0,\infty)$, $p, q > 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 2 - \alpha$, then

$$\int\limits_0^\infty \varphi(x) \left(I_{0+}^{1-\alpha} \psi\right)(x)\, dx = \int\limits_0^\infty \psi(x) \left(I_-^{1-\alpha} \varphi\right)(x)\, dx$$

(see Section 2.5.1 in [18]). We will write this in the notation

$$\left\langle \varphi, I_{0+}^{1-\alpha} \psi \right\rangle = \left\langle \psi, I_-^{1-\alpha} \varphi \right\rangle.$$

In particular, if $\psi = u'$, where $u \in \mathcal{S}(0,\infty)$, then

$$\left\langle \varphi, \mathbb{D}^{(\alpha)} u \right\rangle = \left\langle u', I_-^{1-\alpha} \varphi \right\rangle. \tag{13}$$

It is known [16] that $I_-^{1-\alpha}$ acts continuously from $\Phi_+^{1-\alpha}$ onto $\Phi_+^{\alpha-1}$. Therefore the identity (13) can be used to define $\mathbb{D}^{(\alpha)} u$ as a distribution from $\left(\Phi_+^{1-\alpha}\right)'$, if $u \in C^1(0,\infty)$, and u' has no more than a power-like growth near zero and infinity. This definition agrees with the classical one: if u is continuously differentiable at the origin too, then

$$\begin{aligned} \left\langle u', I_-^{1-\alpha} \varphi \right\rangle &= \frac{1}{\Gamma(1-\alpha)} \int\limits_0^\infty u'(x)\, dx \int\limits_x^\infty (t - x)^{-\alpha} \varphi(t)\, dt \\ &= \frac{1}{\Gamma(1-\alpha)} \int\limits_0^\infty \varphi(t)\, dt \int\limits_0^t (t - x)^{-\alpha} u'(x)\, dx = \left\langle \varphi, \mathbb{D}^{(\alpha)} u \right\rangle, \end{aligned}$$

where $\mathbb{D}^{(\alpha)} u$ is understood in the sense of (9).

A typical example of a function from $\Phi_+^{1-\alpha}$ is the function

$$\varkappa_\alpha(x) = x^{\alpha-2} \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2}\log x\right).$$

It is clear that $\varkappa_\alpha \in \mathcal{S}_+$. Next, if

$$\varkappa(x) = \exp\left(-\frac{\log^2 x}{4}\right) \sin\left(\frac{\pi}{2}\log x\right),$$

then we can write explicitly the Mellin transform

$$\widetilde{\varkappa}(z) = \int\limits_0^\infty x^{z-1}\varkappa(x)\,dx.$$

Namely, by the formula (4.133.1) from [5],

$$\widetilde{\varkappa}(z) = 2\int\limits_0^\infty e^{-t^2/4}\sinh(zt)\sin\frac{\pi t}{2}\,dt = 2\sqrt{\pi}e^{z^2-\frac{\pi^2}{4}}\sin\pi z. \qquad (14)$$

In particular,

$$\int\limits_0^\infty \varkappa_\alpha(x)x^{1-\alpha-k}\,dx = \widetilde{\varkappa}(-k) = 0, \quad k = 0,1,2,\dots,$$

so that indeed $\varkappa_\alpha \in \Phi_+^{1-\alpha}$.

In order to have a full concept of a class of distributions, one needs a result regarding density of a space of test functions in some space of integrable functions. This gives a one-to-one correspondence between ordinary functions and distributions they generate. Here we present such a result though it will not be used directly in this paper. For similar properties in other situations see [17].

Proposition 1. *The space $\Phi_+^{1-\alpha}$ is dense in $L_p((0,\infty),t^{-1}dt)$, $1 \le p < \infty$.*

P r o o f. Let $R = \int\limits_0^\infty \varkappa_\alpha(t)t^{-1}dt$. We have

$$R = \widetilde{\varkappa}(\alpha-2) = 2\sqrt{\pi}\exp\left((\alpha-2)^2 - \frac{\pi^2}{4}\right)\sin\pi\alpha > 0.$$

Denote

$$z_N(x) = \frac{N}{R}\varkappa_\alpha(x^N), \quad N = 1, 2, \dots .$$

Then

$$\int_0^\infty z_N(x)x^{-1}dx = \frac{1}{R}\int_0^\infty \varkappa_\alpha(t)t^{-1}dt = 1. \tag{15}$$

Suppose that $f \in L_p((0,\infty), t^{-1}dt)$. Consider the so-called Mellin convolution

$$(z_N *_M f)(t) = \int_0^\infty z_N(\tau) f\left(\frac{t}{\tau}\right)\tau^{-1}d\tau,$$

that is actually the convolution on the multiplicative group $(0,\infty)$ (note that $t^{-1}dt$ is a Haar measure on that group). Obviously, $z_N *_M f \in \Phi_+^{1-\alpha}$. Denote by $\|\cdot\|_p$ the norm in $L_p((0,\infty), t^{-1}dt)$.

Using (15) we can write

$$(z_N *_M f)(t) - f(t) = \int_0^\infty z_N(\tau)\left[f\left(\frac{t}{\tau}\right) - f(t)\right]\tau^{-1}d\tau.$$

By the generalized Minkowski inequality,

$$\|z_N *_M f - f\|_p \le \int_0^\infty |z_N(\tau)| \left\{\int_0^\infty \left|f\left(\frac{t}{\tau}\right) - f(t)\right|^p t^{-1}dt\right\}^{1/p} d\tau.$$

Next we use the L_p-continuity of shifts on the multiplicative group $(0,\infty)$ (see [10], Theorem 20.4). For any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\left\{\int_0^\infty \left|f\left(\frac{t}{\tau}\right) - f(t)\right|^p t^{-1}dt\right\}^{1/p} < \varepsilon,$$

if $|\tau - 1| < \delta$. Thus,

$$\|z_N *_M f - f\|_p \le 2\|f\|_p \int_{|\tau-1|\ge\delta} |z_N(\tau)|\tau^{-1}d\tau + \varepsilon \int_{|\tau-1|<\delta} |z_N(\tau)|\tau^{-1}d\tau. \tag{16}$$

Note that

$$\int_0^\infty |z_N(\tau)|\tau^{-1}d\tau = \frac{N}{R}\int_0^\infty \left|\varkappa_\alpha(x^N)\right| x^{-1}dx = \frac{1}{R}\int_0^\infty |\varkappa_\alpha(t)|\, t^{-1}dt = C_1$$

where the constant C_1 does not depend on N. On the other hand,

$$|z_N(x)| \le C_2 N x^{N(\alpha-2)} \exp\left(-\frac{N^2\log^2 x}{4}\right).$$

If $|x-1| \ge \delta$, then $\log^2 x \ge b > 0$, so that

$$\int_{1+\delta}^{\infty} |z_N(x)| x^{-1} dx \le C_3 e^{-\frac{N^2 b}{4}} (1+\delta)^{N(\alpha-2)} \to 0,$$

as $N \to \infty$, and (for $\delta < 1$)

$$\int_0^{1-\delta} |z_N(x)| x^{-1} dx \le C_2 N \int_{-\infty}^{\log(1-\delta)} \exp\left\{Nt(\alpha-2) - \frac{N^2 t^2}{4}\right\} dt$$

$$= C_2 \int_{-\infty}^{N\log(1-\delta)} \exp\left\{s(\alpha-2) - \frac{s^2}{4}\right\} ds \to 0,$$

as $N \to \infty$.

As a result, we see that, if N is large enough, the first summand in (16) does not exceed $2\|f\|_p\varepsilon$, while the second $\le C_1\varepsilon$. Thus,

$$\|z_N *_M f - f\|_p \longrightarrow 0, \quad \text{as } N \to \infty.$$

■

2.3. On ratios of the Gamma functions. We will often use the function

$$\rho(t) = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \quad -1 < t < \infty. \tag{17}$$

Here we collect some of its properties.

If $t > \alpha - 1$, the integral representation

$$\frac{1}{\rho(t)} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} e^{-st} e^{(\alpha-1)s} \left(1 - e^{-s}\right)^{\alpha-1} ds \tag{18}$$

holds (see Chapter 4 in [14]). It follows from (18) that the function $t \mapsto \dfrac{1}{\rho(t)}$ is strictly monotone decreasing and $\dfrac{1}{\rho(t)} \to 0$, as $t \to \infty$. Since $\Gamma(t+1-\alpha)$ has a pole at $t = \alpha - 1$, it is seen from (17) that $\rho(t) \to 0$, as $t \to \alpha - 1 + 0$.

If $-1 < t < \alpha - 1$, then (by a well-known identity for the Gamma function)

$$\rho(t) = \frac{\Gamma(-t+\alpha)}{\Gamma(-t)} \cdot \frac{\sin \pi(t+1)}{\sin \pi(t+1-\alpha)}.$$

The integral representation for the ratio of the Gamma functions [14] leads, after an elementary investigation, to the conclusion that $\rho(t)$ is strictly monotone increasing from $-\infty$ to 0.

Thus, we conclude that on the interval $(-1, \infty)$ the function $\rho(t)$ is strictly monotone increasing from $-\infty$ to ∞. The inverse function $\gamma(\lambda)$ solving the equation

$$\rho(t) = \lambda, \quad \lambda \in \mathbb{R},$$

is a well-defined continuous function. Note that $\rho(0) = \dfrac{1}{\Gamma(1-\alpha)}$, so that $\gamma(\lambda) \geq 0$, if $\lambda \geq \dfrac{1}{\Gamma(1-\alpha)}$.

It is known ([14], Chapter 4) that

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} \sim t^{a-b}(1 + O(t^{-1})), \quad t \to \infty, \tag{19}$$

if $b > a$. In particular,

$$\rho(t) \sim t^{\alpha}, \quad \text{as } t \to \infty.$$

For the sequence $\beta(n)$ defined in (10) and appearing in the definition of the Gelfond-Leontiev generalized differentiation operator $\mathfrak{D}_{\alpha}$, we have $\beta(n) = \rho(\alpha n)$, so that the above asymptotics implies the relation

$$\beta(n) \sim C n^{\alpha}, \quad n = 0, 1, 2, \dots . \tag{20}$$

3. α-Entire solutions

Let us consider α-entire solutions of the equation (2) with $a(t) = A(t^{\alpha})$, where A is a polynomial of degree $m \geq 0$. We assume the initial condition $u(0) = u_0$.

THEOREM 1. *Under the above assumptions, the solution $u(t)$ of the equation (2) has the form $u(t) = v(t^{\alpha})$, where v is an entire function whose order does not exceed $(1+m)/\alpha$.*

P r o o f. Seeking the function v, we have $(\mathfrak{D}_\alpha v)(z) = a(z)v(z)$. Let us apply the operator $\mathfrak{I}_\alpha$ (see (11)) to both sides of this equality. We get from (11) and (12) that

$$v(Re^{i\theta}) - v(0) = \frac{e^{i\theta}}{\alpha\Gamma(\alpha)} \int_0^R \left[\left(\frac{R}{r}\right)^{1/\alpha} - 1\right]^{\alpha-1} a(re^{i\theta})v(re^{i\theta})\,dr,$$

which implies the inequality

$$|v(Re^{i\theta})| \le |v(0)| + C\int_0^R \left[\left(\frac{R}{r}\right)^{1/\alpha} - 1\right]^{\alpha-1} |a(re^{i\theta})||v(re^{i\theta})|\,dr$$

(here and below we denote by the same letter C various positive constants).

We have the asymptotic relations

$$t^{\frac{1}{\alpha}} - 1 \sim \frac{1}{\alpha}(t-1), \quad \text{as } t \to 1+0;$$

$$\left(t^{\frac{1}{\alpha}} - 1\right)^{\alpha-1} \sim t^{\frac{\alpha-1}{\alpha}}, \quad \text{as } t \to \infty.$$

Therefore

$$\left(t^{\frac{1}{\alpha}} - 1\right)^{\alpha-1} \le C(t-1)^{\alpha-1}t^{2-\frac{1}{\alpha}-\alpha}, \quad t \ge 1,$$

so that

$$|v(Re^{i\theta})| \le |v(0)| + CR^{2-\frac{1}{\alpha}-\alpha}\int_0^R (R-r)^{\alpha-1}r^{\frac{1}{\alpha}-1}|a(re^{i\theta})||v(re^{i\theta})|\,dr.$$

Since $a(z)$ is a polynomial of degree m, we get

$$|v(Re^{i\theta})| \le |v(0)| + CR^{2-\frac{1}{\alpha}-\alpha}\int_0^R (R-r)^{\alpha-1}r^{\frac{1}{\alpha}-1+m}|v(re^{i\theta})|\,dr,$$

where C does not depend on R, θ. Fixing θ and denoting

$$w(r) = \frac{|v(re^{i\theta})|}{r^{2-\frac{1}{\alpha}-\alpha}},$$

we come to the inequality

$$w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}-\alpha}} + C\int_0^R (R-r)^{\alpha-1} r^{1-\alpha+m} w(r)\, dr. \tag{21}$$

Now we are in a position to apply Henry's theorem (see Lemma 7.1.2 from [9]), which states that the inequality (21) implies the inequality

$$w(R) \leq \frac{|v(0)|}{R^{2-\frac{1}{\alpha}-\alpha}} \mathcal{E}_{\alpha,2-\alpha+m}(CR)$$

where $\mathcal{E}_{\alpha,\sigma}(s)$ is a certain function admitting the estimate

$$\mathcal{E}_{\alpha,\sigma}(s) \leq C s^{\frac{1}{2}\left(\frac{\alpha+\sigma-1}{\alpha}-\sigma\right)} \exp\left(\frac{\alpha}{\alpha+\sigma-1} s^{\frac{\alpha+\sigma-1}{\alpha}}\right).$$

Thus,

$$|v(Re^{i\theta})| \leq C\exp\left(\mu R^{\frac{1+m}{\alpha}}\right)$$

for some $\mu \geq 0$, as desired. ∎

4. Regular singularity

4.1. Formal and α-analytic solutions. Let us consider systems of equations of the form

$$t^\alpha \left(\mathrm{D}^\alpha_{0+} u\right)(t) = A(t^\alpha)u(t), \tag{22}$$

where

$$A(z) = A_0 + \sum_{m=1}^\infty A_m z^m,$$

A_m are $n \times n$ complex matrices and

$$\|A_m\| \leq M\mu^m \ (\mu > 0), \quad m = 0, 1, 2, \dots.$$

Suppose we have a formal series

$$u(t) = \sum_{k=0}^\infty u_k t^{\alpha k}, \quad u_k \in \mathbb{C}^n. \tag{23}$$

Let us substitute the series (23) formally into (22). We get, in accordance with (6), that

$$\sum_{k=0}^{\infty} \beta(k) u_k t^{\alpha k} = \sum_{m,k=0}^{\infty} A_m u_k t^{\alpha(m+k)}$$

where $\beta(k)$ is the sequence (10). Collecting and comparing the terms we find that

$$\beta(l) u_l = \sum_{k=0}^{l} A_k u_{l-k}, \quad l = 0, 1, 2, \dots,$$

or, equivalently,

$$A_0 u_0 = \frac{1}{\Gamma(1-\alpha)} u_0; \tag{24}$$

$$[A_0 - \beta(l)]\, u_l = -\sum_{k=1}^{l} A_k u_{l-k}, \quad l \geq 1. \tag{25}$$

It is natural to call the formal series (23) *a formal solution* of the system (22) if the relations (24), (25) hold.

PROPOSITION 2. *If a formal series (23) is a formal solution of the system (22), then the series (23) is absolutely convergent on some neighbourhood of the origin.*

P r o o f. It follows from (19) that

$$\| [A_0 - \beta(l)]^{-1} \| \leq C l^{-\alpha}, \quad l \geq l_0.$$

In particular, we may assume that

$$\| [A_0 - \beta(l)]^{-1} \| \leq 1, \quad l \geq l_0.$$

Considering, if necessary, $\lambda u(t)$ instead of $u(t)$, with $|\lambda|$ small enough, we may assume that $\|u_0\| \leq 1$.

Let us choose so big $r > 0$ that $\|u_l\| \leq r^l$ for $l \leq l_0$ and

$$M \sum_{k=1}^{\infty} \left(\frac{\mu}{r}\right)^k \leq 1.$$

Then

$$\|u_l\| \leq r^l \quad \text{for all } l.$$

Indeed, if this inequality is proved up to some value of $l \geq l_0$, then

$$\|u_{l+1}\| \leq \left\| \sum_{k=1}^{l+1} A_k u_{l+1-k} \right\| \leq M \sum_{k=1}^{l+1} \mu^k r^{l+1-k} = M r^{l+1} \sum_{k=1}^{l+1} \left(\frac{\mu}{r} \right)^k \leq r^{l+1},$$

and the above inequality implying local convergence in (23) has been proved.

■

The above arguments remain valid for systems of the form

$$t^\alpha \left(\mathbb{D}^{(\alpha)} u \right)(t) = A(t^\alpha) u(t). \tag{26}$$

The only difference is that, instead of (24), we get the relation $A_0 u_0 = 0$, just as in the classical case (see [7]).

4.2. Model scalar equations. Consider the equation

$$t^\alpha \left(\mathrm{D}^\alpha_{0+} \varphi \right)(t) = \lambda \varphi(t), \quad \lambda \in \mathbb{R}. \tag{27}$$

By the relation (6), a solution of the equation (27) is const $\cdot t^{\gamma(\lambda)}$, where γ (the inverse function to ρ) was defined in Section 2.3. For example, if $\lambda = 0$, then we have $\gamma(0) = \alpha - 1$.

If we consider an equation similar to (27), but with the Caputo-Dzhrbashyan derivative, that is

$$t^\alpha \left(\mathbb{D}^{(\alpha)} \varphi \right)(t) = \lambda \varphi(t), \tag{28}$$

then the constant function is a solution of (28) for $\lambda = 0$. Suppose that $\lambda \neq 0$, and φ is a solution of (28), that is $\varphi \in C[0,T]$, the function

$$\left(I^{1-\alpha}_{0+} \varphi \right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \varphi(\tau)\, d\tau$$

is absolutely continuous, and (28) is satisfied with

$$\left(\mathbb{D}^{(\alpha)} \varphi \right)(t) = \frac{d}{dt} \left(I^{1-\alpha}_{0+} \varphi \right)(t) - \frac{1}{t^\alpha \Gamma(1-\alpha)} \varphi(0).$$

We have

$$\left(I^{1-\alpha}_{0+} \varphi \right)(t) = \frac{t^{1-\alpha}}{\Gamma(1-\alpha)} \int_0^1 (1-s)^{-\alpha} \varphi(st)\, ds,$$

and since φ is continuous,

$$\left(I_{0+}^{1-\alpha}\varphi\right)(t) \longrightarrow 0, \quad \text{as } t \to +0.$$

It is known [18] that in these circumstances $I_{0+}^{\alpha} \mathrm{D}_{0+}^{\alpha} \varphi = \varphi$. Note also that I_{0+}^{α} transforms the function $t^{-\alpha}$ into the constant $\Gamma(1-\alpha)$. Dividing the equation (28) by t^{α} and applying I_{0+}^{α} to both sides, we find that

$$\varphi(t) - \varphi(0) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau)\, d\tau$$

$$= \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} \varphi(ts)\, ds.$$

Passing to the limit, as $t \to 0$, and taking into account the continuity of φ, we obtain the identity

$$\frac{\lambda\varphi(0)}{\Gamma(\alpha)} \int_0^1 (1-s)^{-1+\alpha} s^{-\alpha} ds = 0,$$

whence $\varphi(0) = 0$.

Thus, for $\lambda \neq 0$, the equation (28) is equivalent to (27), if (27) is considered for continuous functions vanishing at the origin. The power solution $Ct^{\gamma(\lambda)}$ belongs to this class, if $\lambda > \frac{1}{\Gamma(1-\alpha)}$. It may be instructive to see these solutions, satisfying the equation

$$\varphi(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-\tau)^{-1+\alpha} \tau^{-\alpha} \varphi(\tau)\, d\tau,$$

as examples of non-uniqueness of solutions of linear Volterra integral equations occurring due to the singularity of a kernel.

4.3. Systems with good spectrum. Let us consider the equation (3) with $A(z) = \sum\limits_{m=0}^{\infty} A_m z^m$ where A_m are complex $n \times n$ matrices, the matrix A_0 is Hermitian, and the series converges on a neighbourhood of the origin. Without restricting generality, we may assume that

$$A_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

Following the classical method (see, for example, [1]) we look for a matrix-valued solution (*a fundamental solution*) of the equation (3), in the form

$$u(t) = S(t^\alpha)\psi(t) \tag{29}$$

where $\psi(t) = \operatorname{diag}\left(t^{\gamma(\lambda_1)}, \dots, t^{\gamma(\lambda_n)}\right)$, $S(z) = \sum\limits_{\nu=0}^{\infty} \sigma_\nu z^\nu$, σ_ν $(\nu \geq 1)$ are some unknown matrices, $\sigma_0 = I$.

We have

$$u(t) = \sum_{\nu=0}^{\infty} \sigma_\nu \operatorname{diag}\left(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}\right),$$

whence

$$t^\alpha \left(\mathrm{D}_{0+}^\alpha u\right)(t) = \sum_{\nu=0}^{\infty} \sigma_\nu R_\nu \operatorname{diag}\left(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}\right), \tag{30}$$

where

$$R_\nu u = \operatorname{diag}\left(\frac{\Gamma(\gamma(\lambda_1)+\alpha\nu+1)}{\Gamma(\gamma(\lambda_1)+\alpha\nu+1-\alpha)}, \dots, \frac{\Gamma(\gamma(\lambda_n)+\alpha\nu+1)}{\Gamma(\gamma(\lambda_n)+\alpha\nu+1-\alpha)}\right).$$

On the other hand,

$$A(t^\alpha)u(t) = \sum_{\nu=0}^{\infty} \left(\sum_{m=0}^{\nu} A_m \sigma_{\nu-m}\right) \operatorname{diag}\left(t^{\gamma(\lambda_1)+\alpha\nu}, \dots, t^{\gamma(\lambda_n)+\alpha\nu}\right). \tag{31}$$

Note that $R_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_n) = A_0$, and since $\sigma_0 = I$, the coefficients corresponding to $\nu = 0$ in (30) and (31) coincide. Comparing the rest of the coefficients, we obtain the following system of equations for the matrices σ_k:

$$\sigma_k R_k - A_0 \sigma_k = \sum_{l=0}^{k-1} A_{k-l}\sigma_l, \quad k \geq 1. \tag{32}$$

For each k, the matrix equation (32) for σ_k has a unique solution if the spectra of the matrices R_k and A_0 are disjoint (see Appendix A.1 in [1]), that is

$$\frac{\Gamma(\gamma(\lambda_i)+\alpha k+1)}{\Gamma(\gamma(\lambda_i)+\alpha k+1-\alpha)} \neq \lambda_j \tag{33}$$

for all $i, j \in \{1, \dots, n\}$, or, equivalently, since the left-hand side of (33) equals $\rho(\gamma(\lambda_i)+\alpha k)$,

$$\gamma(\lambda_j) - \gamma(\lambda_i) \neq \alpha k, \quad \text{for all } i, j \in \{1, \dots, n\}.$$

We call our system (3) *a system with good spectrum*, if

$$\gamma(\lambda_j)-\gamma(\lambda_i)\notin \alpha\mathbb{N}, \quad \text{for all } i,j\in\{1,\dots,n\}. \tag{34}$$

This definition extends the classical one [1], since for $\alpha=1$ we would have $\rho(t)=\gamma(t)=t$, and the condition (34) would mean that the eigenvalues of A_0 must not differ by a natural number.

THEOREM 2. *If a system (3) has a good spectrum, then it possesses a fundamental solution (29) where the series for $S(z)$ has a positive radius of convergence.*

P r o o f. By the asymptotic relation (19),

$$\frac{\Gamma(\gamma(\lambda_j)+\alpha k+1)}{\Gamma(\gamma(\lambda_j)+\alpha k+1-\alpha)}\sim(\alpha k)^{\alpha}\left(1+O(k^{-1})\right), \quad k\to\infty,$$

for all $j=1,\dots,n$. Therefore

$$(\alpha k)^{-\alpha}R_k=I+O(k^{-1}), \quad k\to\infty. \tag{35}$$

Let us divide both sides of the equation (32) by $(\alpha k)^{\alpha}$. The resulting equation, considered as a system of scalar equations for n^2 elements of the matrix σ_k, has the coefficients bounded in k and the determinant, which is different from zero for each k and tends to 1, as $k\to\infty$. This implies the estimate

$$\|\sigma_k\|\le ak^{-\alpha}\left\|\sum_{l=0}^{k-1}A_{k-l}\sigma_l\right\|, \quad k\ge 1, \tag{36}$$

where the constant $a>0$ does not depend on k. It follows from (35), (36), and the convergence near the origin of the power series for $A(z)$ that

$$\|\sigma_k\|\le a_1k^{-\alpha}\sum_{l=0}^{k-1}b^{k-l}\|\sigma_l\|, \quad k\ge 1,$$

where a_1 and b are positive constants independent of k.

Define a sequence $\{s_k\}_0^{\infty}$ of positive numbers, setting $s_0=1$,

$$s_k=a_1k^{-\alpha}\sum_{l=0}^{k-1}b^{k-l}s_l, \quad k\ge 1.$$

The induction on k yields the inequality $\|\sigma_k\| \le s_k$ for all $k \ge 0$. On the other hand,

$$\begin{aligned} s_{k+1} &= a_1(k+1)^{-\alpha}\sum_{l=0}^{k} b^{k+1-l}s_l \\ &= \frac{(k+1)^{-\alpha}}{k^{-\alpha}}\left[a_1k^{-\alpha}\left(b\sum_{l=0}^{k-1} b^{k-l}s_l + bs_k\right)\right] \\ &= \frac{(k+1)^{-\alpha}}{k^{-\alpha}}\left(bs_k + a_1k^{-\alpha}bs_k\right) = \frac{(k+1)^{-\alpha}b}{k^{-\alpha}}\left(1+a_1k^{-\alpha}\right)s_k. \end{aligned}$$

Therefore

$$\frac{s_k}{s_{k+1}} \longrightarrow b^{-1}, \quad \text{as } k\to\infty. \tag{37}$$

It follows from (37) that the series $\sum\limits_{k=0}^{\infty} s_k z^k$ has the convergence radius b^{-1} (see Section 2.6 in [21]). Moreover, the series $\sum\limits_{\nu=0}^{\infty} \sigma_\nu z^\nu$ converges for $|z| < b^{-1}$. ■

For the equation (26), a similar construction is valid, if we assume that $\lambda_1, \dots, \lambda_n \ge \dfrac{1}{\Gamma(1-\alpha)}$.

5. Irregular singularity: An example

5.1. A formal solution. In this section we construct a solution, in a sense to be specified, of the equation (5). Looking at classical first order equations, corresponding formally to $\alpha = 1$, we have to consider the equation $t^2y'(t) = \lambda y(t)$ whose solution is $y(t) = \exp(-\lambda t^{-1})$. Therefore it is natural to seek a solution of the equation (5) in the form

$$u(t) = \sum_{n=0}^{\infty} c_n t^{-n\alpha}, \quad c_n \in \mathbb{C}. \tag{38}$$

A fractional derivative of any term in (38) with $n > \alpha^{-1}$ does not make sense classically. However we may apply the distribution theory from Section 2.2. Below we understand the fractional derivative $\mathbb{D}^{(\alpha)}$ in the sense of (13).

PROPOSITION 3. (i) *If* $\mu < 0$, $\mu \neq -1, -2, \ldots$, *then*

$$\mathbb{D}^{(\alpha)}\, t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} t^{\mu-\alpha}. \tag{39}$$

(ii) *If k is a natural number, then*

$$\mathbb{D}^{(\alpha)}\, t^{-k} = \frac{(-1)^{k-1}}{(k-1)!\Gamma(-k+1-\alpha)} t^{-k-\alpha} \log t. \tag{40}$$

P r o o f. Let $\varphi \in \Phi_{+}^{1-\alpha}$. By (13),

$$\left\langle \mathbb{D}^{(\alpha)}\, t^{\mu}, \varphi(t) \right\rangle = \mu \left\langle t^{\mu-1}, \left(I_{-}^{1-\alpha} \varphi \right)(t) \right\rangle. \tag{41}$$

It is clear that the right-hand side of (41) is an entire function of μ. For $\mu > 0$, by virtue of (7),

$$\left\langle t^{\mu-1}, \left(I_{-}^{1-\alpha} \varphi \right)(t) \right\rangle = \left\langle I_{0+}^{1-\alpha} t^{\mu-1}, \varphi(t) \right\rangle = \frac{\Gamma(\mu)}{\Gamma(\mu+1-\alpha)} \left\langle t^{\mu-\alpha}, \varphi(t) \right\rangle.$$

For $\mu < 0$, $\mu \neq -1, -2, \ldots$, the analytic continuation gives the equality (39).

Next, consider the entire function

$$F(\mu) = \langle t^{\mu-\alpha}, \varphi(t) \rangle, \quad \mu \in \mathbb{C}.$$

Note that $F(-k) = 0$, $k \in \mathbb{N}$, by the definition of the space $\Phi_{+}^{1-\alpha}$. We have

$$F'(\mu) = \left\langle \log t \cdot e^{(\mu-\alpha)\log t}, \varphi \right\rangle.$$

In particular,

$$F'(-k) = \int_0^{\infty} t^{-\alpha-k} \log t \cdot \varphi(t)\, dt.$$

As μ belongs to a small neighbourhood of the point $-k$, $F(\mu) = F'(-k)(\mu+k) + o(\mu+k)$. Since the residue of $\Gamma(\mu+1)$ at $\mu = -k$ equals $\dfrac{(-1)^{k-1}}{(k-1)!}$ (see Section 4.4.1 of [20]), we see that the function $\Gamma(\mu+1)F(\mu)$ is holomorphic at $\mu = -k$ (in fact, it is entire) and

$$\Gamma(\mu+1)F(\mu)\big|_{\mu=-k} = \frac{(-1)^{k-1}}{(k-1)!} \int_0^{\infty} t^{-\alpha-k} \log t \cdot \varphi(t)\, dt.$$

Now the equality (39) implies (40). ∎

Returning to (38), we will formally apply $\mathbb{D}^{(\alpha)}$ termwise and find the coefficients c_n comparing resulting terms in (5). It is clear from the equality (40) that such a procedure would fail if some of the numbers $n\alpha$ are integers. Thus, we have to assume that α *is irrational*. Using (39) we find that (formally)

$$t^{2\alpha}\left(\mathbb{D}^{(\alpha)}u\right)(t)=\sum_{n=1}^{\infty}c_n\frac{\Gamma(-n\alpha+1)}{\Gamma(-n\alpha+1-\alpha)}t^{-n\alpha+\alpha}.$$

Substituting this into (5) we come to the recurrence relation

$$c_{n+1}=\lambda\frac{\Gamma(1-(n+2)\alpha)}{\Gamma(1-(n+1)\alpha)}c_n,\quad n\geq 0,$$

and it is easy to find by induction that

$$c_n=\lambda^n\frac{\Gamma(1-(n+1)\alpha)}{\Gamma(1-\alpha)}c_0.$$

Thus, we have found a formal solution

$$u(t)=\frac{c_0}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\Gamma(1-(n+1)\alpha)t^{-n\alpha}\tag{42}$$

of the equation (5). Using the identity

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

we can rewrite (42) in the form

$$u(t)=\frac{c_0\pi}{\Gamma(1-\alpha)}\sum_{n=0}^{\infty}\frac{1}{\sin(\pi(n+1)\alpha)}\frac{t^{-n\alpha}}{\Gamma((n+1)\alpha)}.\tag{43}$$

5.2. The convergence problem. The convergence of the series (43) depends on the arithmetic properties of the irrational number α. It was shown by Hardy [6] that α can be chosen in such a way (to be well approximated by rational numbers) that the series (43) would diverge for small values of t.

An irrational number $\alpha \in (0,1)$ is said to be *poorly approximated by rational numbers*, if there exist such $\varepsilon > 0$, $c > 0$ that for any rational number $\frac{p}{q}$, $p, q \in \mathbb{N}$,

$$\left|\alpha - \frac{p}{q}\right| \geq cq^{-2-\varepsilon}. \tag{44}$$

By the Thue-Siegel-Roth theorem (see [19]) such are all algebraic numbers.

The first statement of the next theorem is actually contained already in the paper [6].

THEOREM 3. *If α is poorly approximated by rational numbers, then the series (43) converges for any $t > 0$. However this series diverges in the space of distributions $\left(\Phi_+^{1-\alpha}\right)'$.*

P r o o f. It follows from (44) that

$$|q\alpha - p| \geq cq^{-1-\varepsilon} \quad \text{for any } p \in \mathbb{N},$$

so that

$$\operatorname{dist}((n+1)\alpha, \mathbb{Z}_+) \geq c(n+1)^{-1-\varepsilon},$$

and taking $l \in \mathbb{Z}_+$, such that $|(n+1)\alpha - l| \leq \frac{1}{2}$, we find that

$$\begin{aligned} |\sin(\pi(n+1)\alpha)| = |\sin(\pi((n+1)\alpha - l)| \geq 2|(n+1)\alpha - l| \\ \geq 2\operatorname{dist}((n+1)\alpha, \mathbb{Z}_+) \geq 2c(n+1)^{-1-\varepsilon}. \end{aligned}$$

Using the Stirling formula we obtain that the series (43) converges for each $t > 0$.

To prove the second assertion, consider $\langle t^{-\alpha n}, \varkappa_\alpha(t)\rangle$, where the function $\varkappa_\alpha$ was defined in Section 2.2. We have, by (14), that

$$\begin{aligned} \left\langle t^{-\alpha n}, \varkappa_\alpha(t)\right\rangle = \int_0^\infty t^{-\alpha n}\varkappa_\alpha(t)\,dt = \widetilde{\varkappa_\alpha}(1-\alpha n) = \widetilde{\varkappa}(\alpha - 1 - \alpha n) \\ = 2\sqrt{\pi}\exp\left((\alpha - 1 - \alpha n)^2 - \frac{\pi^2}{4}\right)\sin(\pi(\alpha - 1 - \alpha n)). \end{aligned}$$

Now we can give a lower estimate of the coefficients in the series (43) understood in the distribution sense: there are such $a, a_1, b > 0$ that

$$\begin{aligned}
&\left|\frac{1}{\sin(\pi(n+1)\alpha)}\cdot\frac{\langle t^{-n\alpha},\varkappa_\alpha(t)\rangle}{\Gamma((n+1)\alpha)}\right| \\
&\qquad \geq a(n+1)^{-b(n+1)}|\sin(\pi\alpha(n-1))|\exp(\alpha^2n^2-2\alpha(\alpha-1)n) \\
&\qquad \geq a_1(n+1)^{-b(n+1)-1-\varepsilon}\exp(\alpha^2n^2-2\alpha(\alpha-1)n) \\
&\qquad = a_1\exp\left\{\alpha^2n^2-2\alpha(\alpha-1)n-(bn+b+1+\varepsilon)\log(n+1)\right\}\to\infty,
\end{aligned}$$

as $n\to\infty$. Therefore the series (43) does not converge in $\left(\Phi_+^{1-\alpha}\right)'$. ■

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* *Institute of Mathematics,*
National Academy of Sciences of Ukraine,
Tereshchenkivska 3,
Kiev, 01601 UKRAINE
e-mail: kochubei@i.com.ua

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